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## QUASI-LINEAR THEORY VIA THE CUMULANT EXPANSION APPROACH

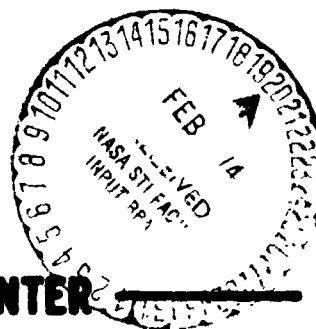
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### ABSTRACT

We use the cumulant expansion technique of Kubo to derive an integro-differential equation for  $\langle f \rangle$ , the average one particle distribution function for particles being accelerated by electric and magnetic fluctuations of a general nature. For a very restricted class of fluctuations, the  $\langle f \rangle$  equation degenerates exactly to a differential equation of Fokker-Planck type. Quasi-linear theory, including the adiabatic assumption, is an exact theory for this limited class of fluctuations. For more physically realistic fluctuations, however, quasi-linear theory is at best approximate. The nature of the approximation is seen by explicitly comparing the usual quasi-linear diffusion coefficient  $D_0$  (which is second order in the fluctuations) to the fourth order contribution  $D_2$ . Quasi-linear theory is asymptotic in the ratio of the wave-particle coherence time  $\tau_c$  to the fluctuation induced deviation time  $\tau_d$  or deflection time  $\tau_D$ . Since  $\tau_c$  varies over phase space, quasi-linear theory may be valid for some regions of phase space and invalid for others.

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## QUASI-LINEAR THEORY VIA THE CUMULANT EXPANSION APPROACH

### 1. INTRODUCTION

Jokipii and Lerche (1973) have recently studied the statistical behavior of charged particles in a random magnetic field of one special type. They find that a differential equation of Fokker-Planck type exactly describes the evolution of the ensemble average one particle distribution function  $\langle f \rangle$ . They compare the exact equation with the equation obtained by making the quasi-linear approximation. The equations are identical provided that the adiabatic approximation is also made on the equation derived via quasi-linear theory.

This Jokipii and Lerche work shows that for their special model, at least, quasi-linear theory is a very good approximation. Is the Jokipii-Lerche result a consequence of their model or is it considerably more general?

We answer this question by first deriving in Section 2, from the stochastic Vlasov Equation the general integro-differential equation which  $\langle f \rangle$  obeys for arbitrarily varying (in space and time) electromagnetic fluctuations. We do so by using the cumulant expansion technique of Kubo (1962, 1963). We then determine in Section 3 the conditions which must be imposed on the fluctuations in order that  $\langle f \rangle$  exactly satisfy a Fokker-Planck or diffusion equation. The conditions are so restrictive that no physically realistic model of fluctuations is included.

In Section 4 we examine the quasi-linear approximation (Drummond and Pines, 1962, 1964; Vedenov et al., 1961, 1962; Bernstein and Engelmann, 1966; Aamodt and Drummond, 1964; Frieman and Rutherford, 1964; Rogister and Oberman, 1969; Kennel and Engelmann, 1966; Galeev and Karpman, 1963; Galeev et al., 1965; Hall and Sturrock, 1967) to the general  $\langle f \rangle$  equation in the context of the cumulant expansion. The quasi-linear approximation is shown to be a short coherence time approximation. The ratio of the neglected terms to the quasi-linear terms depends on the ratio of the coherence time  $\tau_c$  to the deviation time  $\tau_d$  or the deflection time  $\tau_D$ , whichever is shorter. The coherence time is the time required for the random component of the force on a given particle to become uncorrelated; the deviation time is the time required for the random force to significantly alter the particle's position; and the deflection time is the time for this same force to significantly alter the particle's velocity. If the condition  $\tau_c \ll \tau_d, \tau_D$  prevails, quasi-linear theory describes the evolution of  $\langle f \rangle$  very well. When  $\tau_c$  is greater than either  $\tau_d$  or  $\tau_D$ , however, we do not expect quasi-linear theory to be at all accurate.

## 2. STOCHASTIC VLASOV EQUATION

We consider the one particle distribution function  $f(t, x, v)$  that satisfies the Vlasov Equation for particles which interact with a force  $F$ .

$$\frac{\partial f}{\partial t} + (L_o + L_1) f = 0 \quad (1)$$

where

$$L_o \equiv v \cdot \frac{\partial}{\partial x} + F_o \cdot \frac{\partial}{\partial v}$$

is the non-stochastic Vlasov operator and  $L_1 \equiv F_1 \cdot \partial/\partial v$  contains the stochastic part of the force (in general,  $\langle L_1 \rangle \neq 0$ ). We assume that  $L_o$  is constant in time although  $L_1$  can, in general, be time dependent.

Transforming to the interaction representation we have (Kubo, 1963)

$$\frac{\partial g}{\partial t} + \Omega(t) g = 0 \quad (2)$$

where

$$g(t) \equiv e^{L_o t} f(t)$$

and

$$\Omega(t) = e^{L_o t} L_1(t) e^{-L_o t}.$$

Equation 2 may be formally integrated by means of the time ordered exponential operator

$$g(t) = T \exp \left\{ - \int_0^t \Omega(t') dt' \right\} g(0). \quad (3)$$



In Equation 3 the time ordering operator  $T$  orders the  $\Omega$  operators denoted by the exponential such that the operator at the earliest time operates first, the next earliest second and so on, i.e.

$$\begin{aligned}
 T \exp \left\{ - \int_0^t \Omega(t') dt' \right\} = & 1 + \int_0^t \Omega(t_1) dt_1 \\
 + & \int_0^t dt_1 \int_0^{t_1} dt_2 \Omega(t_1) \Omega(t_2) + \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \Omega(t_1) \Omega(t_2) \Omega(t_3) \\
 + & \dots \dots \dots
 \end{aligned} \tag{4}$$

As long as the  $T$  operator stands to the left of an operator expression the operators to its right may be treated as if they were commuting variables or c numbers in any algebraic manipulations. This is so because the  $T$  operator contains and preserves the information about how the operators are to be ordered after the manipulations have been completed.

Equation 3 may now be averaged over an ensemble of  $L_1$ 's to obtain

$$\langle g \rangle(t) = \left\langle T \exp \left\{ - \int_0^t \Omega(t') dt' \right\} \right\rangle_{g_0} \tag{5}$$

where we have chosen  $g_0 \equiv g(0)$  to be uniform over the ensemble. It has been shown by Kubo (1962) that the averaged operator in Equation 5 may be written as a general cumulant expansion

$$\left\langle T \exp \left\{ - \int_0^t \Omega(t') dt' \right\} \right\rangle = \tag{6}$$

$$= T \exp \left\{ \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n \langle \Omega(t_1) \Omega(t_2) \dots \Omega(t_n) \rangle_c \right\} \quad (6 \text{ Cont'd})$$

where the brackets  $\langle \rangle_c$  refer to the cumulant average, i. e.

$$\langle A \rangle_c = \langle A \rangle; \langle AB \rangle_c = \langle AB \rangle - \langle A \rangle \langle B \rangle \text{ etc.}$$

(Kubo, 1962).

The time ordering operator  $T$  applies to everything to its right including the expansions of the cumulant average  $\langle \rangle_c$  in terms of the conventional averages  $\langle \rangle$ . For the cumulants with  $n \geq 3$  the  $T$  operator notation becomes essential when expressing cumulant averages in terms of conventional averages. This can be seen as follows

$$\begin{aligned} \langle ABC \rangle_c &\equiv \langle ABC \rangle - \langle A \rangle \langle BC \rangle - \langle B \rangle \langle AC \rangle \\ &\quad - \langle AB \rangle \langle C \rangle + 2 \langle A \rangle \langle B \rangle \langle C \rangle. \end{aligned}$$

One will note that in order to express all of the required correlated averages using the conventional notation it was impossible to maintain the ordering,  $ABC$ , in all of the terms. An operator such as  $T$  standing to the left that tells in what order the various operators are to be applied removes this notational difficulty.

If we introduce the notation

$$K_n(t) \equiv \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \dots \int_0^{t_{n-1}} dt_n \langle \Omega(t_1) \Omega(t_2) \Omega(t_3) \dots \Omega(t_n) \rangle_c$$

we may write

$$\langle g \rangle(t) = T \exp \left\{ \sum_{n=1}^{\infty} (-1)^n K_n(t) \right\} g_0. \quad (7)$$

In an attempt to find a kinetic equation for  $\langle g \rangle$  we differentiate Equation 7 with respect to time, noting that the time derivative commutes with  $T$ , to obtain

$$\frac{\partial \langle g \rangle}{\partial t} = T \left[ \sum_{n=1}^{\infty} (-1)^n \frac{\partial K_n}{\partial t} \right] \exp \left\{ \sum_{n=1}^{\infty} (-1)^n K_n \right\} g_0 . \quad (8)$$

This appears to be as far as we can go maintaining full generality. We must now examine what exact conditions the statistical operators  $\Omega(t)$  must satisfy in order that Equation 8 be equivalent to a Fokker-Planck equation or what approximate conditions they must satisfy in order that Equation 8 may be approximated by a Fokker-Planck equation.

### 3. CONDITIONS FOR EXACT KINETIC EQUATIONS TO EXIST

We note first of all that the operator  $\partial K_n(t)/\partial t$  is an  $n^{\text{th}}$  order differential operator. If we wish to obtain a kinetic equation of finite order the series in square brackets in Equation 8 must terminate at some finite  $n$  in general and at  $n = 2$  in particular if we wish a diffusion or Fokker-Planck type equation.

The only way to obtain this termination for all values of  $t$  is to require that all cumulant averages of the statistical operator  $\Omega(t)$  higher than the second vanish identically. This is equivalent to requiring  $\Omega(t)$  to be a Gaussian process. It thus appears that the first requirement that the operator  $\Omega(t)$  must satisfy is that it be a Gaussian process; in so far as cumulant averages of  $\Omega$  to powers higher than the second are non-vanishing the kinetic equation will deviate from a Fokker-Planck type equation by the inclusion of derivative operators of order greater than the second.

If we now assume that  $\Omega(t)$  represents a Gaussian process Equation 8 becomes

$$\begin{aligned} \frac{\partial \langle g \rangle}{\partial t} = T \left[ -\langle \Omega(t) \rangle + \int_0^t \langle \Omega(t) \Omega(\tau) \rangle_c d\tau \right] \\ \times \exp \{ -K_1 + K_2 \} g_0 . \end{aligned} \quad (9)$$

Since  $t$  is the latest time in the expression the  $\langle \Omega(t) \rangle$  in the square brackets is already time ordered and may thus be brought to the left of the  $T$  operator.

So doing and using Equation 7 gives

$$\frac{\partial \langle g \rangle}{\partial t} + \langle \Omega(t) \rangle \langle g \rangle = T \int_0^t d\tau \langle \Omega(t) \Omega(\tau) \rangle_c \exp\{-K_1 + K_2\} g_0. \quad (10)$$

Since the variable  $\tau$  in the integral above ranges over all times we may not, in general, commute the  $\langle \Omega(t) \Omega(\tau) \rangle_c$  and T operations to obtain a kinetic equation for  $\langle g \rangle$ . The effect of the T operator in Equation 10 is to mix the  $\Omega(\tau)$  operator with every term of the expansion of  $\exp\{-K_1 + K_2\}$  and keep us from having a clearly separate operator applied to  $\langle g \rangle$ .

The only circumstance under which Equation 10 becomes a purely differential equation for  $\langle g \rangle$  is when the T operator is unnecessary. This occurs when the order in which the  $\Omega$  operators are applied is unimportant. This can be so only if we have

$$[\Omega(t_1), \Omega(t_2)] = 0, t_1 \neq t_2 \quad (11)$$

the  $\Omega$  operators must commute with themselves at different times.

We have thus arrived at the second condition that must be met by the statistical operator  $\Omega(t)$ . We shall see immediately that these two conditions, Gaussian statistics and commutivity at different times, are sufficient to guarantee that  $\langle f \rangle$ , the ensemble averaged distribution function, satisfies exactly a Fokker-Planck type kinetic equation. However, upon closer investigation the requirement of commutivity, though simply expressed in Equation 11, will turn out to impose stringent requirements on the physical systems.

If Equation 11 is satisfied we may disregard the T operator in Equation 10 to obtain

$$\frac{\partial \langle g \rangle(t)}{\partial t} + \langle \Omega(t) \rangle \langle g \rangle(t) = \int_0^t \langle \Omega(t) \Omega(\tau) \rangle_c d\tau \langle g \rangle(t) \quad (12)$$

Leaving the interaction representation, we may express Equation 12 in terms of  $\langle f \rangle$  as

$$\left( \frac{\partial}{\partial t} + L_0 + \langle L_1(t) \rangle \right) \langle f \rangle(t) = \int_0^t \langle L_1(t) e^{L_0(\tau-t)} L_1(\tau) e^{-L_0(\tau-t)} \rangle_c d\tau \langle f \rangle(t). \quad (13)$$

The operator  $L_1(t)$  represents the effect of the fluctuating force  $F_1$  at time  $t$  at a phase space point  $x, v$ . The operator  $e^{L_0(\tau-t)} L_1(\tau) e^{-L_0(\tau-t)}$  represents the effect of this force at time  $\tau$  and at the point in phase space where a particle at  $x, v$  at time  $t$  would have been at time  $\tau$  if it were undergoing the unperturbed motion generated by the operator  $L_0$ . If the statistics of  $L_1$  are stationary in time (which we shall assume hereafter) and if there is a time  $\tau_c$  such that  $L_1(t)$  and  $e^{L_0(\tau-t)} L_1(\tau) e^{-L_0(\tau-t)}$  become uncorrelated for  $t-\tau > \tau_c$ , the integral in Equation 13 becomes independent of  $t$  for  $t \gg \tau_c$  and our kinetic equation becomes a Fokker-Planck equation with constant coefficients. Note that this loss of correlation can come about either through an intrinsic time variation in  $L_1(t)$  or through the propagator  $e^{L_0(\tau-t)}$  removing the particles a distance over which the fluctuating force is spatially uncorrelated or both.

Turning to the conditions that must be met by a system such that Equation 13 is an exact kinetic equation for  $\langle f \rangle$ , we can say little about the assumption of Gaussian statistics. It is a common assumption, partly because the central limit theorem (Feller, 1968) states that a tendency towards Gaussian statistics is a common occurrence, but mainly because of its mathematical tractability which most other types of statistics do not share. The requirement that  $\Omega(t_1)$  and  $\Omega(t_2)$  commute for  $t_1 \neq t_2$ , on the other hand, has some rather severe implications.

One way of fulfilling this requirement is to demand  $\Omega(t_1) = \Omega(t_2)$  for all  $t_1, t_2$ . This implies

$$\frac{\partial \Omega}{\partial t} = e^{L_0 t} L_0 L_1 e^{-L_0 t} + e^{L_0 t} \frac{\partial L_1}{\partial t} e^{-L_0 t} - e^{L_0 t} L_1 L_0 e^{-L_0 t} = 0. \quad (14)$$

Multiplying from the left with  $e^{-L_0 t}$  and from the right with  $e^{L_0 t}$  yields

$$\frac{\partial L_1}{\partial t} + [L_0, L_1] = 0 \quad (14')$$

or in other words  $L_1$  is a constant of the motion generated by  $L_0$ .

A physical model of such a system could probably be constructed, but one suspects that such a model would be rather artificial. Also note that in this case the integral in Equation 12 or Equation 13 will grow linearly in time so that a time-diverging diffusion coefficient results.

Another way of fulfilling the commutivity requirement is to have

$$[L_0, L_1(t)] = 0 \text{ for all } t \quad (15)$$

and

$$[L_1(t_1), L_1(t_2)] = 0 \text{ for all } t_1, t_2. \quad (16)$$

Most models discussed to date are of this type. Expressing  $L_0$  and  $L_1$  in conventional variables  $L_0 = \mathbf{v} \cdot \partial/\partial \mathbf{x} + \mathbf{F}_0 \cdot \partial/\partial \mathbf{v}$  and  $L_1 = \mathbf{F}_1 \cdot \partial/\partial \mathbf{v}$  we see that Equations 15 and 16 imply

$$\left( \mathbf{v} \cdot \frac{\partial \mathbf{F}_1}{\partial \mathbf{x}} + \mathbf{F}_0 \cdot \frac{\partial \mathbf{F}_1}{\partial \mathbf{v}} - \mathbf{F}_1 \cdot \frac{\partial \mathbf{F}_0}{\partial \mathbf{v}} \right) \cdot \frac{\partial}{\partial \mathbf{v}} - \mathbf{F}_1 \cdot \frac{\partial}{\partial \mathbf{x}} = 0 \quad (15')$$

$$\left( \mathbf{F}_1(t_1) \cdot \frac{\partial \mathbf{F}_1(t_2)}{\partial \mathbf{v}} - \mathbf{F}_1(t_2) \cdot \frac{\partial \mathbf{F}_1(t_1)}{\partial \mathbf{v}} \right) \cdot \frac{\partial}{\partial \mathbf{v}} = 0. \quad (16')$$

The above expressions are quite complex and no general solutions have been found but one can make some general statements about them. First of all the coefficients of the velocity derivatives must vanish. One could merely require that the distribution function have no velocity gradients in these directions but distributions functions that are unbounded in velocity space are extremely unphysical. Fortunately this is not quite so true for spatially unbounded systems or we would be forced to require  $\mathbf{F}_1 = 0$  and the entire problem would evaporate. One can therefore require spatial homogeneity to the extent that is needed to keep  $\mathbf{F}_1 \cdot \partial/\partial \mathbf{x} = 0$ . This homogeneity will be more or less isotropic depending on to what extent, if any,  $\mathbf{F}_1$  is constrained in direction.



The coefficient of the velocity gradient in Equation 15' can be considered as a partial differential equation for a general velocity and space dependent force. However, if we limit our discussion to electromagnetic forces the velocity dependence is specified and terms of different orders in  $v$  must be set equal to zero separately. This gives

$$\mathbf{E}_0 \times \mathbf{B}_1 = \mathbf{E}_1 \times \mathbf{B}_0 \quad (17)$$

$$\mathbf{v} \cdot (\nabla \mathbf{E}_1 + \mathbf{B}_1 \mathbf{B}_0 - \mathbf{B}_0 \mathbf{B}_1) = 0 \quad (18)$$

$$\mathbf{v} \cdot \nabla \mathbf{B}_1 \times \mathbf{v} = 0 \quad (19)$$

Since the velocity is a free parameter and generally not constrained in magnitude or direction the coefficients of  $v$  must vanish order by order. Thus

$$\nabla \mathbf{E}_1 + \mathbf{B}_1 \mathbf{B}_0 - \mathbf{B}_0 \mathbf{B}_1 = 0 \quad (18')$$

$$\nabla \mathbf{B}_1 = 0 \quad (19')$$

Similarly from Equation 16', it follows that

$$\mathbf{E}_1(t_1) \times \mathbf{B}_1(t_2) = \mathbf{E}_1(t_2) \times \mathbf{B}_1(t_1) \quad (20)$$

$$\mathbf{B}_1(t_1) \times \mathbf{B}_1(t_2) = 0 \quad (21)$$

We may add the full set of Maxwell's Equations to Equations 17, 18', 19', 20 and 21 above and ask what arrangement of fields are then allowed. It is straightforward to verify the following:

- a.  $\mathbf{B}_1$  must be parallel to  $\mathbf{B}_0$ ;

- b.  $B_1$  must be spatially homogeneous and constant in time;
- c.  $E_1$  must be spatially homogeneous and only the component of  $E_1$  that is parallel to  $B_0$  may vary in time.  $E_1$  thus represents an infinite wavelength plasma oscillation in which the plasma current exactly cancels the displacement current.

This case of one dimensional plasma oscillations was treated by Lerche (1972), who assumed Gaussian statistics for the fluctuating electric field and derived an exact equation of the form of Equation 13. For this completely spatially homogeneous situation  $L_0 = 0$ ,  $L_1 = (q E_1(t)/m) \cdot \partial/\partial v$ ,  $\langle I \rangle = 0$  and from Equation 13

$$\frac{\partial \langle f \rangle}{\partial t} = \frac{q^2}{m^2} \int_0^t d\tau \langle E_1(t) E_1(\tau) \rangle \frac{\partial^2 \langle f \rangle}{\partial v^2} \quad (22)$$

where  $\langle E_1(t) E_1(\tau) \rangle$  is here the correlation function for the fluctuating electric field.

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Lerche has compared his exact result, his analogue of Equation 22, with the time asymptotic diffusion equation derived by quasi-linear theory. He finds that the diffusion coefficients in the two equations differ by a factor of 2, a result in contradiction with the later conclusions of Jokipii and Lerche (1973) and with our own findings in the next section. We have found in Lerche's original work an algebraic error which accounts for the discrepancy.

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While this elementary system is the only one in which both the commutivity of  $\Omega(t_1)$  and  $\Omega(t_2)$  and Maxwell's Equations are rigorously satisfied, further latitude is available if we require the commutivity of  $\Omega(t_1)$  and  $\Omega(t_2)$  by fields which only approximately satisfy Maxwell's Equations. The model considered by Jokipii and Lerche (1973) falls into this latter category.

Jokipii and Lerche consider a situation in which the only superposed electromagnetic field is a time fluctuating, spatially homogeneous magnetic component  $\hat{e}_z B_1(t)$  ( $E_0 = E_1 = B_0 = 0$ ). The induced electric field required by Maxwell's Equations  $\nabla \times E_1 = -c^{-1} \partial B_1 / \partial t$  is assumed small and negligible, although it is clear, as the authors admit, that  $E_1$  can not be small everywhere in the infinite space. With the additional assumption that spatial variations be only along  $B_1$  so that  $\nabla = \hat{e}_z \partial / \partial z$ ,  $L_0 = v_z \partial / \partial z$ ,  $L_1 = \omega_1(t) \partial / \partial \theta$ , and Equations 15' and 16' are satisfied. Here  $\omega_1$  is the cyclotron frequency  $q B_1 / mc$  and  $\theta = \tan^{-1} v_y / v_x$  is the gyrophase. In the main body of their paper, Jokipii and Lerche take Gaussian statistics for  $B_1$ , assume  $\langle B_1 \rangle = 0$ , and come up with the identical result

$$\frac{\partial \langle f \rangle}{\partial t} + v_z \frac{\partial \langle f \rangle}{\partial z} = \int_0^t d\tau \langle \omega_1(t) \omega_1(\tau) \rangle_c \frac{\partial^2 \langle f \rangle}{\partial \theta^2} \quad (23)$$

which we obtain from Equation 13.

In their appendix Jokipii and Lerche take a different statistical model for  $B_1$ . Here  $B_1$  has the same initial value  $b_0$  in each realization of the ensemble but otherwise varies randomly such that the probability distribution  $P(B_1, t)$

at a later time satisfies an Ornstein-Uhlenbeck (Uhlenbeck and Ornstein, 1930) equation

$$\tau_c \frac{\partial P}{\partial t} = \frac{\partial}{\partial B_1} (B_1 P) + \langle B_1^2 \rangle \frac{\partial^2 P}{\partial B_1^2} \quad (24)$$

Their assertion is that the statistics of  $B_1$  in this case need not be Gaussian.

However, a quantity whose probability distribution satisfies Equation 24 is a Gaussian process; in fact it is a Markovian, Gaussian process (Doob, 1942) so no violation of our basic conditions is actually found here.

If  $B_1(t)$  is an Ornstein-Uhlenbeck process with initial value  $b_0$  it is a Gaussian process with  $\langle B_1(t) \rangle = b_0 \exp -t/\tau_c$ . The time integral  $\int_0^t B_1(\tau) d\tau$  is also a Gaussian process; Uhlenbeck and Ornstein (1930) give expressions for the mean and mean square of this integral. Recalling that the diffusion coefficient is one half the time derivative of the second cumulant average of  $\int_0^t B_1(\tau) d\tau$  we readily obtain

$$\begin{aligned} & \left( \frac{\partial}{\partial t} + v_z \frac{\partial}{\partial z} - \omega_1(0) e^{-t/\tau_c} \frac{\partial}{\partial \theta} \right) \langle f \rangle(t) \\ &= \langle B_1^2(t \rightarrow \infty) \rangle \tau_c \left( 1 + e^{-2t/\tau_c} - 2e^{-t/\tau_c} \right) \frac{\partial^2}{\partial \theta^2} \langle f \rangle(t). \end{aligned} \quad (25)$$

Equation 25 differs from that of Jokipii and Lerche (1973) by the numerical coefficients on the terms of the diffusion coefficients and the algebraic sign of the first order  $\theta$ -derivative term. We attribute this difference to algebraic errors on their part. Our diffusion coefficient reduces to zero at  $t = 0$ , as it should,

while theirs does not. Also, only the minus sign on the  $\partial/\partial\theta$  term is consistent with a clockwise gyration of electrons as viewed in the direction of  $B_1$ .

We see that the cumulant expansion method very quickly yields kinetic equations when they are exactly derivable, and it further makes quite explicit the rather stringent requirements that must be fulfilled in these cases. We shall now see how the quasi-linear approximation fits into such a scheme and estimate the order of magnitude of the error that is involved.

#### 4. APPROXIMATE KINETIC EQUATIONS

If we return to Equation 10 and insert the appropriate expressions for  $K_1$  and  $K_2$  we may write it as

$$\begin{aligned} \frac{\partial \langle g \rangle}{\partial t} + \langle \Omega(t) \rangle \langle g \rangle = T \int_0^t \langle \Omega(t) \Omega(\tau) \rangle_c d\tau \\ \times \exp \left\{ - \int_0^t d\tau_1 \mathcal{O}(\tau_1) \right\} g_0 \end{aligned} \quad (10')$$

where  $\mathcal{O}(\tau_1)$  is the operator

$$\mathcal{O}(\tau_1) = \langle \Omega(\tau_1) \rangle - \int_0^{\tau_1} d\tau_2 \langle \Omega(\tau_1) \Omega(\tau_2) \rangle_c.$$

We note once again that it is the mixing of the operator  $\Omega(\tau)$  with the exponential as required by the operator  $T$  that prevents us from immediately writing down an exact kinetic equation for  $\langle g \rangle$  and hence for  $\langle f \rangle$ . All of the difficulty arises from that region of the integration over  $\tau_1$  where  $t \geq \tau_1 \geq \tau$  for when  $\tau \geq \tau_1$  the operator  $\Omega(\tau)$  is already properly ordered and  $T$  may be brought through to act directly on the exponential operator.

If it can be shown that the region of the  $\tau_1$  integration where  $t \geq \tau_1 \geq \tau$  is relatively unimportant an approximation may be based on this fact. Such a situation may arise if it can be shown that the operator  $\langle \Omega(t) \Omega(\tau) \rangle_c$  goes to zero if  $\tau$  is sufficiently different from  $t$ . If a "correlation time"  $\tau_c$  exists such that  $\langle \Omega(t) \Omega(\tau) \rangle_c \rightarrow 0$  for  $t - \tau \gg \tau_c$  and if  $t \gg \tau_c$  the significant contribution to

the  $\tau$  integral will come from those values of  $\tau$  within a few correlation times of  $t$ . Hence only for  $t > \tau_1 > t - N\tau_c$ ,  $N = O(1)$  will there be a problem with the  $T$  operator and this region of  $\tau_1$ , being small compared to  $t - N\tau_c > \tau_1 \geq 0$ , may perhaps be neglected.

The effect of the region  $t > \tau_1 > t - N\tau_c$  may be neglected in at least two different ways. The first would be to neglect the effect of the  $T$  operator in this region and move it through to the exponential operator as an approximation. This approach, of course, leads to Equation 13 but this time as an approximation rather than an exact equation. The difficulty with this approach is that it is virtually impossible to tell just what has been neglected and thus estimate the validity of the approximation.

The second method does not suffer from this defect. Note first of all that since the  $T$  operator gives the complete prescription as to how the  $\Omega$  operators are to be ordered in any expansion we may treat them as c-numbers as long as they are to the right of  $T$ . We therefore may write

$$\begin{aligned}
\exp \left\{ - \int_0^t d\tau_1 \mathcal{O}(\tau_1) \right\} &= \exp \left\{ - \int_0^\tau d\tau_1 \mathcal{O}(\tau_1) - \int_\tau^t d\tau_1 \mathcal{O}(\tau_1) \right\} \\
&= \exp \left\{ - \int_0^\tau d\tau_1 \mathcal{O}(\tau_1) \right\} \exp \left\{ - \int_\tau^t d\tau_1 \mathcal{O}(\tau_1) \right\} \\
&= \exp \left\{ - \int_0^\tau d\tau_1 \mathcal{O}(\tau_1) \right\} + \left[ \exp \left\{ - \int_\tau^t d\tau_1 \mathcal{O}(\tau_1) \right\} - 1 \right] \exp \left\{ - \int_0^\tau d\tau_1 \mathcal{O}(\tau_1) \right\}
\end{aligned} \tag{26}$$

The approximation consists of neglecting the second term of the right hand side of Equation 26 and keeping only the first term.

Since  $\Omega(\tau)$  and the retained term of Equation 26 are now properly time ordered we may bring the T operation through to the exponential operator to obtain after some algebra.

$$\left( \frac{\partial}{\partial t} + L_0 + \langle L_1(t) \rangle \right) \langle f \rangle(t) = \int_0^t \langle L_1(t) e^{L_0(\tau-t)} L_1(\tau) \rangle_c \langle f \rangle(\tau) d\tau. \quad (27)$$

It can be readily seen that we have arrived at the standard quasi-linear approximation to the kinetic equation for  $\langle f \rangle$ . The term that we have neglected

$$\begin{aligned} & T \int_0^t \langle \Omega(t) \Omega(\tau) \rangle_c d\tau \left[ \exp \left\{ - \int_{\tau}^t d\tau_1 \mathcal{O}(\tau_1) \right\} - 1 \right] \\ & \times \exp \left\{ - \int_0^{\tau} d\tau_2 \mathcal{O}(\tau_2) \right\} g_0 \end{aligned} \quad (28)$$

may be expanded in a systematic manner by first expanding the exponential in square brackets in a power series and then breaking up the second exponential into two regions of  $\tau_2$  as was done before. This process may be continued indefinitely to produce the standard series although this method of generating it is more difficult than is the standard approach. We shall return shortly to a more detailed analysis of this remainder term.

It is interesting to note that if starting with Equation 27 we make the additional approximation usually referred to as the "adiabatic hypothesis", namely



replace  $\langle f \rangle(\tau)$  by  $\langle f \rangle(t)$  and shield it from the free streaming propagator  $e^{L_0(\tau - t)}$  by placing  $e^{-L_0(\tau - t)}$  in front of it, one obtains Equation 13 that we previously obtained directly by simply neglecting the effect of the  $T$  operator in the region  $\tau_1 > \tau$ . We see therefore that the "adiabatic hypothesis" is on an equal footing with the quasi-linear approximation itself and it would be inconsistent with the spirit of the scheme to accept one and reject the other. In all likelihood the two approximations stand or fall together.

We must now address ourselves to the question of the validity of our neglect of the term given by expression 28. We have assumed that a correlation time  $\tau_c$  exists such that the  $\tau$  integral cuts off for  $t - \tau \gg \tau_c$ . This implies that the integral in the exponent  $\int_{\tau}^t d\tau_1 \mathcal{O}(\tau_1)$  will extend over at most a few times  $\tau_c$  and the hope is that this contribution will be in some sense small. Since  $\mathcal{O}(\tau_1)$  contain derivative operators that operate on the distribution function as well as the force field function the operator can not be bounded in any rigorous sense. To proceed we must therefore make some assumptions about the distribution function. We shall therefore assume that the gradients in phase space of the distribution function are negligible compared to the gradients of the random force function. This is well within the spirit of diffusion theory where one expects to be able to describe the evaluation of a distribution function only on time and length scales that are large compared to the correlation scale of the random forces.

On making this assumption one can now write the right hand side of Equation 10 as a diffusion operator

$$\frac{\partial}{\partial \mathbf{v}} \cdot (\mathbf{D}_0 + \mathbf{D}_1 + \mathbf{D}_2 + \dots) \cdot \frac{\partial}{\partial \mathbf{v}}$$

where  $\mathbf{D}_0$  is the quasi-linear diffusion coefficient and  $\mathbf{D}_1, \mathbf{D}_2$  etc, are correction terms generated by the quasi-linear series. For the sake of simplicity we shall set  $\langle \Omega \rangle = 0$  in the following and compare the terms  $\mathbf{D}_0$  and  $\mathbf{D}_2$  ( $\mathbf{D}_1$  is now  $= 0$ ).

Before continuing we will need to introduce some additional notation. We shall employ the six-dimensional phase space position vector  $\mathbf{z} \equiv (\mathbf{x}, \mathbf{v})$  and a six dimensional force vector  $\mathbf{F}_0 \equiv (\mathbf{v}, \mathbf{F}_0)$ ,  $\mathbf{F}_1 \equiv (0, \mathbf{F}_1)$ . With this notation we have

$$\mathbf{L}_0 = \mathbf{F}_0 \cdot \frac{\partial}{\partial \mathbf{z}} \equiv \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}} + \mathbf{F}_0 \cdot \frac{\partial}{\partial \mathbf{v}}$$

and

$$\mathbf{L}_1 = \mathbf{F}_1 \cdot \frac{\partial}{\partial \mathbf{z}} \equiv \mathbf{F}_1 \cdot \frac{\partial}{\partial \mathbf{v}}.$$

Furthermore since we will need scalar products of quantities that may be separated from each other in an expression we will occasionally make use of a component notation with summation of repeated indices implied, i.e.,

$$\mathbf{F}_0 \cdot \frac{\partial}{\partial \mathbf{z}} \equiv F_{0i} \frac{\partial}{\partial z_i} \equiv \sum_{i=1}^5 F_{0i} \frac{\partial}{\partial z_i}.$$

Recalling that setting  $\langle \Omega \rangle = 0$  makes  $\mathcal{O}(\tau_1) = - \int_0^{\tau_1} d\tau_2 \langle \Omega(\tau_1) \Omega(\tau_2) \rangle$  we may turn to Equation 28 and expand the exponential in square brackets keeping

only the lowest order term to obtain the approximate expression for the neglected term.

$$\begin{aligned}
& T \int_0^t d\tau <\Omega(t)\Omega(\tau)> \int_\tau^t d\tau_1 \int_0^{\tau_1} d\tau_2 <\Omega(\tau_1)\Omega(\tau_2)> \\
& \times \exp \left\{ - \int_0^\tau d\tau_3 \mathcal{O}(\tau_3) \right\} g_0 .
\end{aligned} \tag{28'}$$

We see that for the range  $0 \leq \tau_2 \leq \tau$  the T operator once again mixes  $\Omega(\tau_2)$  with the exponential. We may treat this problem as we did before by neglecting the region between  $\tau_2 - \tau$  in the exponent integral whenever  $\tau_2 < \tau$ .

We then have two terms

$$\begin{aligned}
& T \int_0^t d\tau \int_\tau^t d\tau_1 <\Omega(t)\Omega(\tau)> \left\{ \int_0^\tau d\tau_2 <\Omega(\tau_1)\Omega(\tau_2)> \right. \\
& \times \exp \left[ - \int_0^{\tau_2} d\tau_3 \mathcal{O}(\tau_3) \right] + \int_\tau^{\tau_1} d\tau_2 <\Omega(\tau_1)\Omega(\tau_2)> \\
& \times \exp \left[ - \int_0^\tau d\tau_3 \mathcal{O}(\tau_3) \right] \Big\} g_0 \\
& = T \int_0^t d\tau \int_\tau^t d\tau_1 <\Omega(t)\Omega(\tau)> \left\{ \int_0^\tau d\tau_2 <\Omega(\tau_1)\Omega(\tau_2)> <g>(\tau_2) \right. \\
& \left. + \int_\tau^{\tau_1} d\tau_2 <\Omega(\tau_1)\Omega(\tau_2)> <g>(\tau) \right\} .
\end{aligned} \tag{29}$$

We may now relabel the variables in the first term by interchanging the labels  $\tau$  and  $\tau_2$  and interchange the order of integration to reinstate the original

order to obtain

$$T \int_0^t d\tau \int_\tau^t d\tau_1 \int_\tau^{\tau_1} d\tau_2 \left\{ \langle \Omega(t)\Omega(\tau) \rangle \langle \Omega(\tau_1)\Omega(\tau_2) \rangle \right. \\ \left. + \langle \Omega(t)\Omega(\tau_2) \rangle \langle \Omega(\tau_1)\Omega(\tau) \rangle \right\} \langle g \rangle(\tau). \quad (30)$$

Inserting the appropriate expressions for the  $\Omega$  factor and converting from  $\langle g \rangle$  to  $\langle f \rangle$  we obtain for the second order diffusion coefficient  $D_2$  the expression

$$D_2(t) = \int_0^t d\tau \int_\tau^t d\tau_1 \int_\tau^{\tau_1} d\tau_2 \left\langle F_1(t) e^{-L_0(t-\tau_1)} F_1(\tau_1) \cdot \frac{\partial}{\partial z} \right. \\ \left. e^{-L_0(\tau_1-\tau_2)} F_1(\tau_2) \cdot \frac{\partial}{\partial z} e^{-L_0(\tau_2-\tau)} F_1(\tau) \right\rangle', \quad (31)$$

where the primed brackets means that in taking the average only two of the three possible correlations are included, i. e.,  $\langle t, \tau_1, \tau_2, \tau \rangle' \equiv \langle t, \tau_2 \rangle \langle \tau_1, \tau \rangle + \langle t, \tau \rangle \langle \tau_1, \tau_2 \rangle$  where here the groupings indicate correlations only and not the ordering of the operator.

Expression 31 as it stands is not very transparent and it would be difficult to estimate the relative magnitude of  $D_2$  with respect to  $D_0$  from it in this form. We will therefore transform it to a more recognizable form so that it will be easier to visualize what physical parameters govern the size of  $D_2$  as compared to  $D_0$ .

It will be useful to first review the algebraic properties of the Lie operator  $e^{-L_0 t}$  and the characteristic solution  $z^*(z, t)$  and  $z^0(z, t)$ .  $z^*$  and  $z^0$  are

solutions of the respective differential equations

$$\left( \frac{\partial}{\partial t} + L_0 + L_1 \right) z^*(z, t) = 0 \quad (32)$$

$$\left( \frac{\partial}{\partial t} + L_0 \right) z^0(z, t) = 0 \quad (33)$$

with the common initial value

$$z^*(z, 0) = z^0(z, 0) = z \quad (34)$$

It is readily seen that  $z^*$  and  $z^0$  are the phase space positions at a time  $-t$  of a particle which is moving in the force fields  $F_0 + F_1$  and  $F_0$  respectively and is at the point  $z$  at  $t = 0$ . Equation 33 may be solved by the Lie operator as

$$z^0(z, t) = e^{-L_0 t} z$$

and hence the effect of this operator on the derivative operator is

$$e^{-L_0 t} \frac{\partial}{\partial z} = \frac{\partial}{\partial z^0(t)} e^{-L_0 t} = \frac{\partial z_i}{\partial z^0} \frac{\partial}{\partial z_i} e^{-L_0 t}. \quad (35)$$

And clearly

$$\frac{\partial}{\partial z} e^{-L_0 t} = \frac{\partial z_i^0(t)}{\partial z} e^{-L_0 t} \frac{\partial}{\partial z_i}. \quad (36)$$

It should be further noted that any function of  $z^*(z^0)$  is itself a solution of Equation 32 (Equation 33).

Equation 32 may be cast as an integral equation using the solution to 33,  $z^0$  and the Lie operator as

$$z^*(t, \tau) = z^0(t - \tau) - \int_{\tau}^t L_1 e^{-L_0(t-\tau)} z^*(\tau, \tau) d\tau. \quad (37)$$

This integral equation may be solved by iteration in a straightforward manner.

The result may be expressed in terms of the deviation of the Vlasov trajectory  $z^*$  from the free streaming trajectory  $z^0$  as

$$z^*(t, \tau) - z^0(t - \tau) \equiv \Delta z(t, \tau) = \Delta z^1 + \Delta z^2 + \Delta z^3 + \dots$$

where

$$\Delta z^1(t, \tau) = - \int_{\tau}^t d\tau_1 e^{-L_0(t-\tau_1)} F_1(\tau_1) \cdot \frac{\partial}{\partial z} z^0(\tau_1 - \tau) \quad (38)$$

$$\begin{aligned} \Delta z^2(t, \tau) &= - \int_{\tau}^t d\tau_1 e^{-L_0(t-\tau_1)} F_1(\tau_1) \cdot \frac{\partial}{\partial z} \Delta z^1(\tau_1, \tau) \\ &= \int_{\tau}^t d\tau_1 e^{-L_0(t-\tau_1)} F_1(\tau_1) \cdot \frac{\partial}{\partial z} \int_{\tau}^{\tau_1} d\tau_2 e^{-L_0(\tau_1-\tau_2)} F_1(\tau_2) \cdot \frac{\partial}{\partial z} z^0(\tau_2 - \tau) \end{aligned} \quad (39)$$

etc. We shall return to these equations shortly. If we now return to Equation 31 we note that the expression can be broken into two terms, one proportional to  $\partial F_1(\tau)/\partial z$  and the other proportional to  $\partial^2 F_1(\tau)/\partial z^2$ . Adopting the convention that parentheses bound the range of the Lie operators and the derivative operator and noting that

$$(e^{-L_0 t} A B C \dots) = (e^{-L_0 t} A) (e^{-L_0 t} B) (e^{-L_0 t} C) \dots$$

we obtain

$$\begin{aligned} D_2(t) &= \int_0^t d\tau \int_{\tau}^t d\tau_1 \int_{\tau}^{\tau_1} d\tau_2 \left\langle F_1(t) \left( e^{-L_0(t-\tau_1)} F_1 \cdot \frac{\partial}{\partial z} e^{-L_0(\tau_1-\tau_2)} \right. \right. \\ &\quad \left. \left. \times F_{1i}(\tau_2) \frac{\partial z_j^0(\tau_2 - \tau)}{\partial z_i} \right) \left( e^{-L_0(t-\tau)} \frac{\partial F_1(\tau)}{\partial z_j} \right) \right\rangle' \end{aligned} \quad (40)$$

$$\begin{aligned}
& + \int_0^t d\tau \int_\tau^t d\tau_1 \int_\tau^{\tau_1} d\tau_2 \left\langle F_1(t) \left( e^{-L_o(t-\tau_1)} F_{1i}(\tau_1) \frac{\partial z_j^o(\tau_1-\tau_2)}{\partial z_i} e^{-L_o(\tau_1-\tau_2)} \right. \right. \\
& \quad \left. \left. \times F_{1k}(\tau_2) \frac{\partial z_\ell^o(\tau_2-\tau)}{\partial z_k} \frac{\partial z_m^o(\tau_2-\tau)}{\partial z_j} \right) \left( e^{-L_o(t-\tau)} \frac{\partial^2 F_1(\tau)}{\partial z_\ell \partial z_m} \right) \right\rangle'. \quad (40 \text{ Cont'd.})
\end{aligned}$$

Using the group property of the Lie operator, i.e.,

$$e^{-L_o t_1} z^o(t_2) = e^{-L_o t_1} e^{-L_o t_2} z = e^{-L_o(t_1+t_2)} z = z^o(t_1+t_2)$$

and Equation 35, we may move  $\partial z_m^o(\tau_2-\tau)/\partial z_j$  in the second term to the right through the operator  $e^{-L_o(\tau_1-\tau_2)}$  to obtain for the second term

$$\begin{aligned}
\text{second term} &= \int_0^t d\tau \int_\tau^t d\tau_1 \int_\tau^{\tau_1} d\tau_2 \left\langle F_1(t) \left( e^{-L_o(t-\tau_1)} F_{1i}(\tau_1) \frac{\partial z_j^o(\tau_1-\tau_2)}{\partial z_i} \right. \right. \\
& \quad \left. \left. \frac{\partial z_m^o(\tau_1-\tau)}{\partial z_j^o(\tau_1-\tau_2)} e^{-L_o(\tau_1-\tau_2)} F_{1k}(\tau_2) \frac{\partial z_\ell^o(\tau_2-\tau)}{\partial z_k} \right) \left( e^{-L_o(t-\tau)} \frac{\partial^2 F_1(\tau)}{\partial z_\ell \partial z_m} \right) \right\rangle' \\
&= \int_0^t d\tau \int_\tau^t d\tau_1 \int_\tau^{\tau_1} d\tau_2 \left\langle F_1(t) \left( e^{-L_o(t-\tau_1)} F_{1i}(\tau_1) \frac{\partial z_m^o(\tau_1-\tau)}{\partial z_i} \right) \right. \\
& \quad \left. \left( e^{-L_o(t-\tau_2)} F_{1k}(\tau_2) \frac{\partial z_\ell^o(\tau_2-\tau)}{\partial z_k} \right) \left( e^{-L_o(t-\tau)} \frac{\partial^2 F_1(\tau)}{\partial z_\ell \partial z_m} \right) \right\rangle' \quad (41) \\
&= \frac{1}{2} \int_0^t d\tau \left\langle F_1(t) \left( \int_\tau^t d\tau_1 e^{-L_o(t-\tau_1)} F_1(\tau_1) \cdot \frac{\partial}{\partial z} z_m^o(\tau_1-\tau) \right) \right. \\
& \quad \left. \left( \int_\tau^t d\tau_2 e^{-L_o(t-\tau_2)} F_1(\tau_2) \cdot \frac{\partial}{\partial z} z_\ell^o(\tau_2-\tau) \right) \left( e^{-L_o(t-\tau)} \frac{\partial^2 F_1(\tau)}{\partial z_\ell \partial z_m} \right) \right\rangle'.
\end{aligned}$$

The last step was justified by the symmetry of the integral under the exchange

$$\tau_1 \leftrightarrow \tau_2.$$

Since the integral over  $\tau_1$  and  $\tau_2$  in the first parentheses of Equation 40 is just equal to expression 39 for  $\Delta z^2(t, \tau)$  and the first two parentheses of expression 41 are each equal to  $\Delta z^1(t, \tau)$ , we may write  $D_0$  and  $D_2$  together as

$$D_0 + D_2 = \int_0^t d\tau \left\langle F_1(t) \left[ (F_1(\tau))_{z=z^0(t-\tau)} + \Delta z^2_i(t, \tau) \left( \frac{\partial F_1(\tau)}{\partial z_i} \right)_{z=z^0(t-\tau)} + \frac{1}{2} (\Delta z^1_i(t, \tau) \Delta z^1_j(t, \tau)) \left( \frac{\partial^2 F_1(\tau)}{\partial z_i \partial z_j} \right)_{z=z^0(t-\tau)} \right] \right\rangle'. \quad (42)$$

$D_0$  is evaluated by propagating  $F_1(\tau)$  along the free streaming orbits  $z^0(t-\tau)$  ignoring the effect of  $F_1$  on the orbit. Clearly  $D_2$  is trying to correct this neglect to second order in  $F_1$  via a Taylor's series in the orbit corrections  $\Delta z(t, \tau)$ . This picture is not strictly correct because of the prime in the averaging brackets meaning that one of the terms that would be present in the normal average is not present.

$$D_0 + D_2 + D_4 + \dots \neq \int_0^t d\tau \langle F_1(z, t) F_1(z^*(t-\tau), \tau) \rangle \quad (43)$$

However, the correction terms do appear to be proportional to the trajectory corrections to the appropriate order such that if  $\Delta z = 0$  or if  $F_1$  were not dependent on  $z$ ,  $D_0$  would be exact.



## 5. DISCUSSION

Our assertion in the Introduction that quasi-linear theory is a short coherence time approximation now has clearer meaning. We see from Equation 42 that  $D_2 \ll D_0$  provided that  $|\Delta z \cdot \partial/\partial z F_1|_{rms} \ll |F_1|_{rms}$  for the length of time that contributes significantly to the integrand. For the spatial components of  $z$ ,  $|\Delta z \cdot \partial/\partial z F_1|_{rms} \cong |F_1|_{rms}$  defines the deviation time  $\tau_d$ , that interval over which the average particle's actual position deviates by an amount the order of one correlation length  $x_c$  from its zeroth order position. For the velocity components of  $z$ ,  $|\Delta z \cdot \partial/\partial z F_1|_{rms} \cong |F_1|_{rms}$  defines the deflection time  $\tau_D$ , that interval over which the average  $\Delta v$  becomes of the order of the unperturbed  $v$ . The cut-off time of the integrand is produced by the incoherence of  $F_1(t)$  and  $F_1(\tau)$  for times  $t - \tau > \tau_c$ . Thus  $D_2 \ll D_0$  requires  $\tau_d, \tau_D \gg \tau_c$ .

For those restricted physical systems which satisfy Equations 17-19 and thus lead to an exact Fokker-Planck description of  $\langle f \rangle$ , the fluctuating force  $F_1$  is independent of phase space position and thus both  $\tau_d$  and  $\tau_D$  are infinite.

For realistic physical systems  $\tau_d$  and/or  $\tau_D$  are finite and quasi-linear theory provides at best an approximate description of  $\langle f \rangle$ . For any given system, however, quasi-linear theory may not be a uniformly adequate approximation, since  $\tau_c$  can vary over phase space and become arbitrarily long in certain regions.

Kaiser, Jones, and Birmingham (1973) pointed out this fact in their paper discussing diffusion in the presence of only a static, spatially fluctuating magnetic field  $\hat{e}_z \delta B(x)$ . ( $x, y, z$  are Cartesian coördinates here.) Zeroth order orbits are  $x(\tau) = x(t) + v \cos \theta (\tau - t)$ ,  $v = (v_y^2 + v_x^2)^{1/2} = \text{const.}$  (where  $\tan \theta = v_y/v_x$ ) in this 2 dimensional model. For particles with no initial  $v_x$ , i. e.,  $\cos \theta = 0$ ,  $F_1[x(t)]$  and  $F_1[x(\tau)]$  are correlated for an arbitrarily long time so that  $\tau_c \rightarrow \infty$ . Quasi-linear theory is useless for describing these particles.

This same inadequacy of quasi-linear theory manifests itself in the problem of magnetically induced pitch angle scattering (Hasselmann and Wibberenz, 1968; Kulsrud and Pearce, 1969; Jokipii, 1971, 1972; Klimas and Sandri, 1971, 1973), where the pitch angle  $\theta$  is defined with respect to a uniform average background field  $B_0 \hat{e}_x$ . For this situation  $\Delta \mu \cong \delta B/B_0$  ( $\mu = \cos \theta$ ),  $\Delta x \cong \delta B v t/B_0$  and thus  $\tau_d = B_0 x_c / \delta B v$ . But  $\tau_c = x_c / \mu v$  and hence  $\tau_c / \tau_d = \delta B / \mu B_0$ . If this ratio is to be small,  $\delta B / \mu B_0 \ll 1$ : quasi-linear theory is applicable only to particles which have  $\mu \gg \delta B / B_0$  (Jones, Birmingham, and Kaiser, 1973).

The shortcomings of quasi-linear theory can be improved upon by incorporating some effects of the fluctuations in the lowest order orbits  $z_0$  (Dupree, 1966; Weinstock, 1968, 1969; Birmingham and Bornatici, 1971; Mikhailovskii, 1964; Al'tshul' and Karpman, 1965; Orszag and Kraichnan, 1967; Aamodt, 1967; Gaiev, 1967; Ichimaru, 1970; Rudakov and Tsytovich, 1971; Volk et al., 1973). The fluctuations thus assist in removing particles from correlated regions of

phase space. Jones, Kaiser, and Birmingham (1973) propose a partial averaging procedure and show that for the pitch angle scattering problem ordinary quasi-linear theory is modified by the replacement of  $\mu$  by  $(\mu^2 + \delta B^2/B_0^2)^{1/2}$ . The correlation time  $x_c/v(\mu^2 + \delta B^2/B_0^2)^{1/2}$  is thus finite for all regions of phase space.

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